2023 Precalculus Exam

1. Find $\cos(450°)$.

   (a) 0 (b) 1 (c) -1 (d) 2 (e) $\frac{\sqrt{2}}{2}$

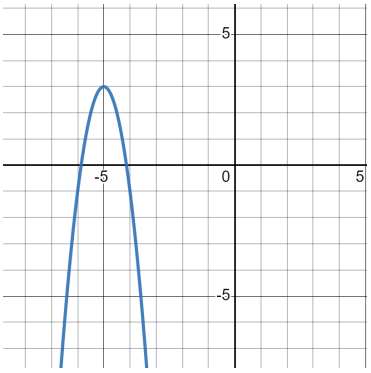
2. If $f_1(x) = x + 2$ and $f_2(x) = 3x^2 + 4x - 5$, which of the following equates to $g(x) = 3x^2 + 16x + 15$?

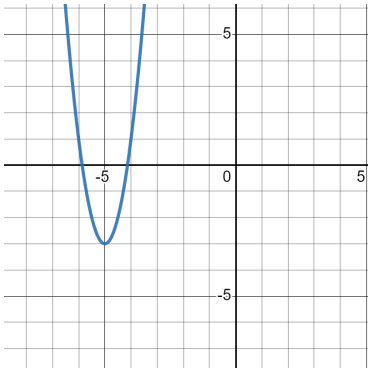
   (a) $g(x) = (f_1 \circ f_2)(x)$
   (b) $g(x) = (f_2 \circ f_1)(x)$
   (c) $g(x) = f_1(x) \cdot f_2(x)$
   (d) $g(x) = f_2(x)/f_1(x)$
   (e) None of the above.

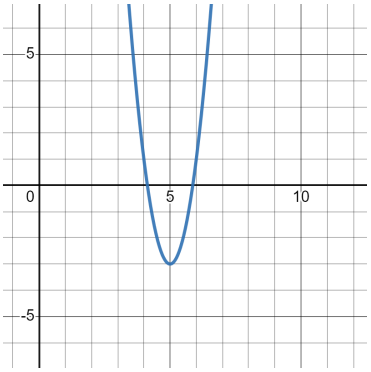
3. Suppose that $f(x) = ax^2 + bx + c$ attains its minimum at $(5, 4)$. Which of the following could be $f(x)$?

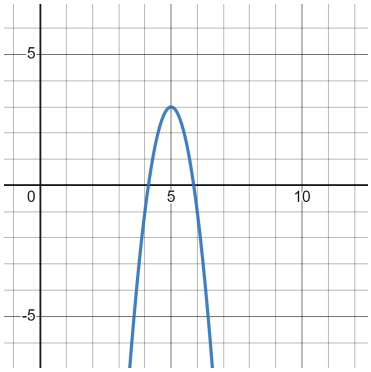
   (a) $f(x) = x^2 - 10x + 29$
   (b) $f(x) = x^2 + 10x + 21$
   (c) $f(x) = x^2 + 10x + 29$
   (d) $f(x) = -x^2 - 10x + 21$
   (e) $f(x) = -x^2 + 10x - 21$

4. Which of the following is the graph of $f(x) = -4x^2 - 40x - 97$?

   (A)
   

   (B)
   

   (C)
   

   (D)
   

   (a) A (b) B (c) C (d) D
   (e) None of the above

5. Solve for x:

$$\ln\left[(x-1)^2\right] - \ln(2) = 1$$

   (a) $x = \sqrt{1 \pm 2e}$ (b) $x = 1 \pm \sqrt{2e}$ (c) $x = 1 \pm 2e$ (d) $x = 1 \pm e\sqrt{2}$
   (e) None of the above

6. Given the function $y = x^2 + 5x + 2$, what can be said about its roots?

   (a) There are two real roots.
   (b) There is only one real root.
   (c) There are only two imaginary roots.
   (d) There are no real roots.
   (e) There is only one imaginary root.

7. What is $\dfrac{6-2i}{8+4i}$ simplified?

   (a) $\dfrac{40-40i}{80}$ (b) $0.5 - 0.5i$ (c) $\dfrac{20i+25}{6}$ (d) $0.25i + 4i$
   (e) None of the above.

8. Assume that $0 < x < \pi$ and $\sin(x) = 3\cos(x)$. What is the value of $\sin(x)\cos(x)$?

   (a) $\dfrac{1}{5}$ (b) $\dfrac{1}{6}$ (c) $\dfrac{3}{10}$ (d) $\dfrac{2}{9}$
   (e) None of the above.

9. **(Tiebreaker #1)** Which of the following is an asymptote for the graph of

$$y = \frac{x^2 - 4x + 3}{x^3 - 2x^2 - 5x + 6} \ ?$$

   (a) $y = 1$ (b) $x = -2$ (c) $x = 3$ (d) $x = -4$
   (e) None of the above.

10. Suppose that $\cos(\theta + \pi) = \sin(\theta)$. What is a possible value for θ that makes this equation true?

    (a) 0 (b) π (c) $\pi/2$ (d) $3\pi/4$
    (e) None of the above.

11. Solve for x:

$$25^{20x-23} = 125^x.$$

    (a) $\dfrac{23}{19}$ (b) 0 (c) $\dfrac{46}{37}$ (d) 5
    (e) No solution exists.

12. Evaluate the expression:

$$\sin(\tan^{-1}(-1))$$

    (a) 1 (b) $\dfrac{\sqrt{3}}{2}$ (c) $-\dfrac{\sqrt{2}}{2}$ (d) $-\dfrac{1}{2}$
    (e) None of the above.

13. A square has three corners at the points $(1,3)$, $(8,6)$ and $(5,13)$. What are the coordinates of its fourth corner?

(a) $(-1,9)$ (b) $(-2,8)$ (c) $(0,9)$ (d) $(-2,10)$ (e) $(-1,10)$

14. **(Tiebreaker #2)** Which of the following numbers is largest?

(a) $5^{\sqrt{5}}$ (b) $\sqrt{5}^{5}$ (c) $\sqrt{5}^{2\sqrt{5}}$ (d) $(2\sqrt{5})^{\sqrt{5}}$ (e) 50

15. What is the period of the function $f(x)$?

$$f(x) = \sqrt[3]{2023^{2023} - \pi}\ \sin\left(4\pi\left(x - \frac{5}{6}\right)\right)$$

(a) $\frac{12\pi}{5}$ (b) 2 (c) $\frac{3}{5}$ (d) $\frac{1}{2}$
(e) None of the above.

16. Suppose that

$$\begin{bmatrix} x & 2y \\ 3y & 4x \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Solve for x and y.

(a) $x = 2$ and $y = 1$.
(b) $x = -\frac{7}{5}$ and $y = \frac{13}{10}$.
(c) $x = \frac{6}{29}$ and $y = \frac{13}{29}$.
(d) $x = \frac{3}{8}$ and $y = \frac{2}{6}$.
(e) None of the above.

17. Suppose that

$$\cot(x) + \tan(x) > 0.$$

What must be true of $\sin(x)$ and $\cos(x)$?

(i) $\sin(x) > 0$ and $\cos(x) > 0$
(ii) $\sin(x) > 0$ and $\cos(x) < 0$
(iii) $\sin(x) < 0$ and $\cos(x) > 0$
(iv) $\sin(x) < 0$ and $\cos(x) < 0$

(a) (i) or (iv). (b) Only (iii). (c) Only (ii). (d) Only (i). (e) Only (iv).

18. Let $\triangle ABC$ be a right triangle such that $\angle A = 90°$, and $\angle B = 30°$. Compute

$$\cot(\angle C)\sec(\angle C)\csc(\angle C).$$

(a) $\frac{4}{3}$ (b) 4 (c) $\frac{1}{4}$ (d) 0 (e) -1

19. What is the length of the shortest possible path from the point $(3,0)$ to the line $y = 2x - 1$?

(a) $\sqrt{2}$ (b) 2 (c) $\sqrt{5}$ (d) 3 (e) $\sqrt{10}$

20. **(Tiebreaker #3)**: Let α be an acute angle. Compute the value of

$$S = \sqrt{\sin^4 \alpha + 4\cos^2 \alpha} + \sqrt{\cos^4 \alpha + 4\sin^2 \alpha}.$$

(a) 1 (b) 2 (c) 3 (d) 4
(e) The value depends on α.

21. How many 4-digit numbers have at least one 5 but no more than three 5's?

(a) 3167 (b) 3168 (c) 9000 (d) 2438
(e) None of the above.

22. Which of the following is the remainder when $x^3 + 2x^2 - 5x - 10$ is divided by $(x - 2)$?

(a) 0 (b) 2 (c) 4 (d) 6
(e) None of the above.

23. A triangle has angles $45°, 60°$, and $75°$. If the side opposite the $45°$ angle has length 2, what is the length of the side opposite the $60°$ angle?

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $\sqrt{5}$ (d) $\sqrt{6}$ (e) $\sqrt{8}$

24. **(Tiebreaker #4)** What is the coefficient of x^2 when the binomial $\left(x^2 - \frac{1}{x}\right)^{10}$ is expanded and simplified?

(a) 10 (b) 81 (c) -120 (d) 120 (e) 210

25. Simplify $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$.

(a) $\sec(2\theta) + \tan(2\theta)$ (b) $\cos(2\theta) - \sin(2\theta)$
(c) $\cos(2\theta) + \sin(2\theta)$ (d) $\cos^2(2\theta) + \sin^2(2\theta)$
(e) $\sec(2\theta) - \tan(2\theta)$

26. Let $ABCD$ be a quadrilateral such that $AB = 7$, $BC = 4$, $CD = 6$, $DA = 5$, and $\angle B = 60°$. What is $\cos(\angle D)$?

(a) $\sqrt{1 - \dfrac{(61 - \sqrt{37})^2}{3600}}$ (b) $\dfrac{2}{5}$ (c) $\dfrac{\sqrt{21}}{5}$ (d) $\dfrac{\sqrt{4}}{5}$
(e) None of the above.

27. Suppose that u, o, and p are all positive integers. How many solutions are there to $u + o + p = 23$?

(a) 231 (b) 552 (c) 462 (d) 276 (e) 300

28. Consider the equation

$$(z^{2023/2} - i)(z^{2023/2} + i) - 2i^{2024} = 0.$$

How many distinct solutions does this equation have within the complex numbers?

(a) 0 (b) 2023 (c) 2024 (d) 2 (e) 1

29. Let $\triangle ABC$ be a right triangle such that $BC = 2$, $\angle A = 15°$, and $\angle B = 90°$. Find AB.

(a) $AB = 3$ (b) $AB = 2 + 2\sqrt{3}$ (c) $AB = 2$ (d) $AB = 4 + 2\sqrt{3}$
(e) $AB = 2 - 2\sqrt{5}$

30. **(Tiebreaker #5)** Recall that given a nonzero complex number z, we can find $r > 0$ and $-\pi < \theta \leq \pi$ such that $z = re^{i\theta}$. The *principal branch* of the complex logarithm of z is defined as

$$\text{Log}(z) = \ln r + i\theta.$$

Suppose that $\text{Log}(z) = \dfrac{\pi}{2} - i\dfrac{\pi}{4}$. Find z.

(a) $\dfrac{e^{\pi/2}}{\sqrt{2}}(1 - i)$ (b) 1 (c) $\dfrac{1}{\sqrt{5}}e^{\pi/6 - i\pi/12}$ (d) $e^{3\pi/2 - 3i\pi/4}$
(e) None of the above